

Regression Analysis: Fundamentals



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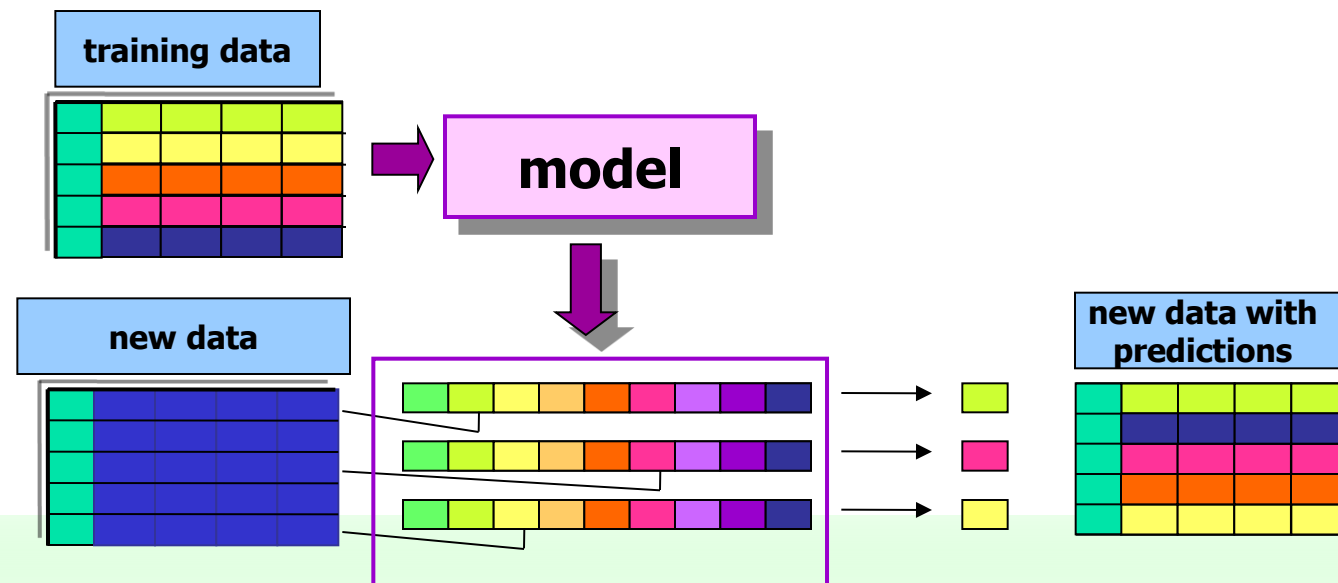
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Introduction to the regression analysis



■ Objectives

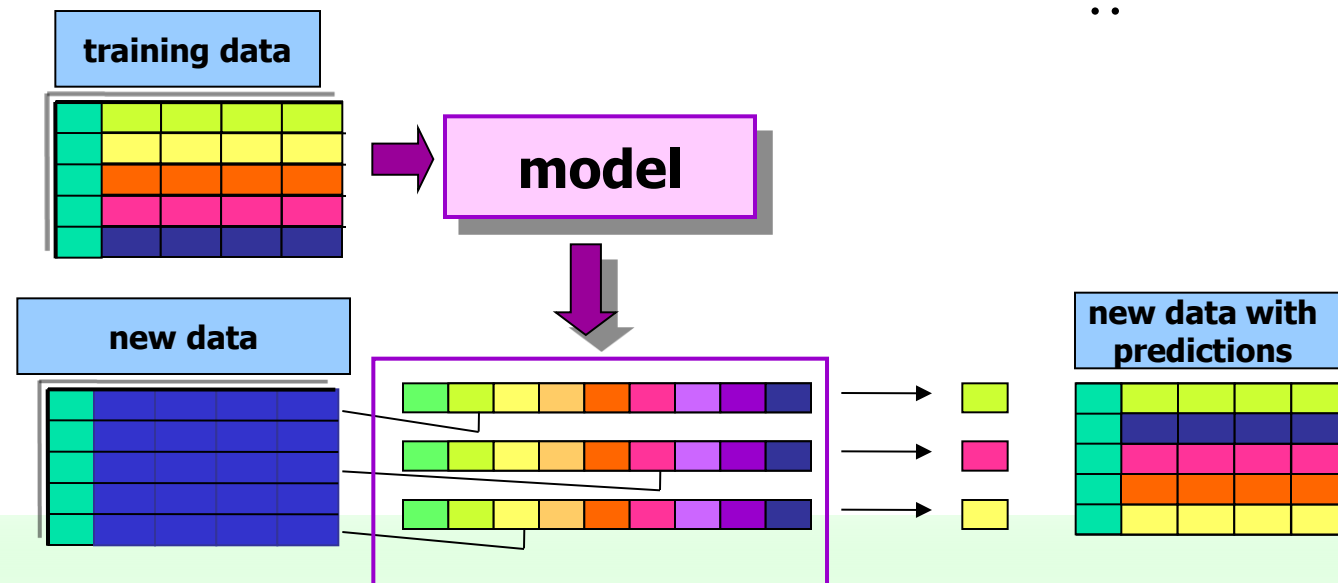
- Prediction of a **numerical target variable**
- Definition of an **interpretable model** of a given phenomenon



Introduction to the regression analysis



- Approach discussed in this set of slides
 - **Linear regression**
 - **SVMs (SVR)**
 - Other approaches
 - k-Nearest Neighbours
 - Decision trees
 - ..

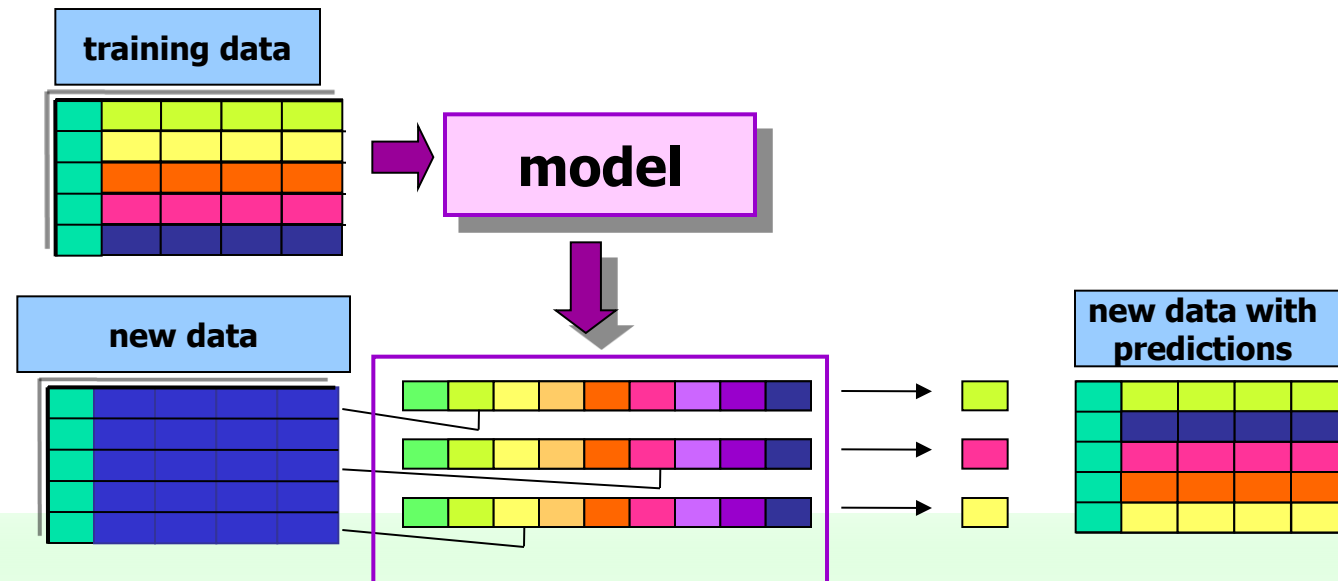


Introduction to the regression analysis



■ Requirements

- **accuracy**
- interpretability
- scalability
- **noise and outlier management**



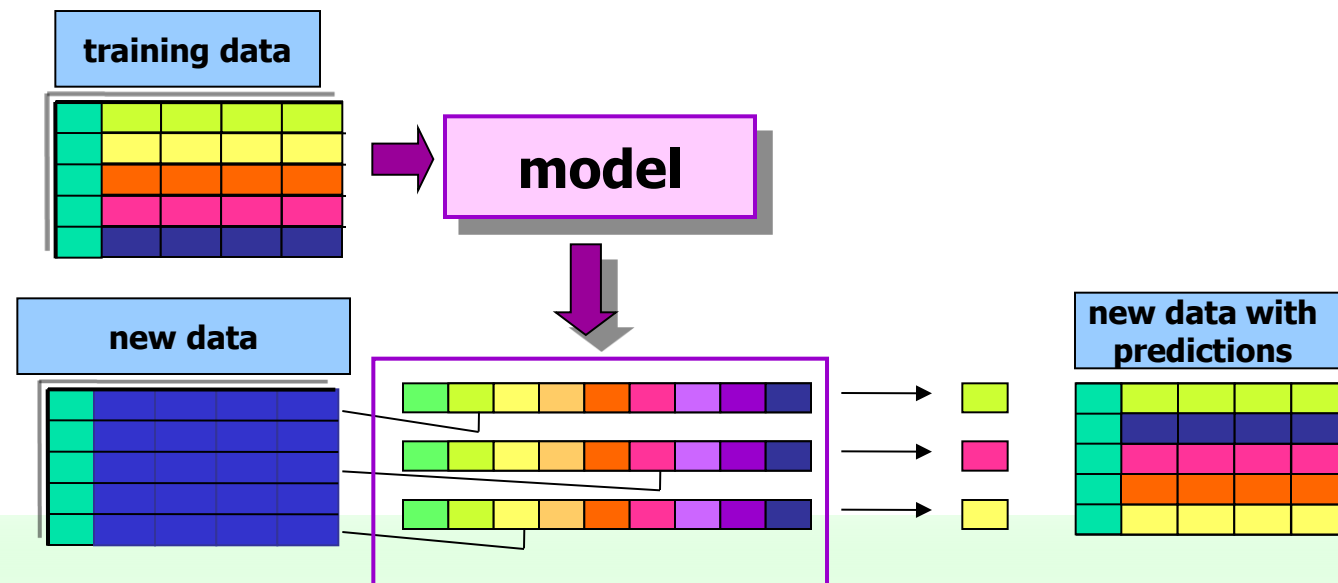
Introduction to the regression analysis



■ Applications

- Estimating the cost of a house
- Estimating the remaining useful life (RUL) of an industrial equipment
- Industrial Vehicle Usage Predictions
- Predicting the Number of Free Floating Car Sharing Vehicles within Urban Areas

■ ...



Introduction to regression



- The term "regression" was coined by **Francis Galton** in 1877 to describe a biological phenomenon
 - the heights of descendants of tall ancestors tend to regress down towards a normal average (i.e., regression toward the mean)
- Father of regression **Carl F. Gauss** (1777-1855)

Definition



- Given
 - A numerical target attribute
 - A collection of data objects also characterized by the target attribute
- The regression task finds a model that allows predicting the target variable value of new objects through
 - $y=f(x_1, x_2, \dots x_n)$

Regression analysis



- Regression analysis can be classified based on
 - **Number of explanatory variables**
 - Simple regression: single explanatory variable
 - Multiple regression: includes any number of explanatory variables
 - **Types of relationship**
 - Linear regression: straight-line relationship
 - Non-linear: implies curved relationships (e.g., logarithmic relationships)
 - **Temporal dimension**
 - Cross Sectional: data gathered from the same time period
 - Time Series: involves data observed over equally spaced points in time

Linear regression



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Simple linear regression



$$y = \beta_0 + \beta_1 x$$

- The regression line provides an **interpretable model** of the phenomenon under analysis
 - y : **estimated** (or predicted) **value**
 - β_0 : estimation of the **regression intercept**
 - The intercept represents the estimated value of y when x assumes 0
 - β_1 : estimation of the **regression slope**
 - x : **independent variable**

Simple linear regression



$$y = \beta_0 + \beta_1 x$$

- *Least squares method*

- β_0 and β_1 can be obtained by **minimizing the Residual sum of squares (RSS)** that is the sum of the squared residuals
 - differences between actual values (y) and estimated ones (\hat{y})

$$\begin{aligned} \min RSS &= \min \sum_i (y_i - \hat{y}_i)^2 = \\ &= \min \sum_i (y_i - (\beta_0 + \beta_1 x_i))^2 \end{aligned}$$

Estimation of the parameters by least squares



$$y = \beta_0 + \beta_1 x$$

$$\beta_1 = \frac{\sum_i (x_i - \bar{x})(y_i - \bar{y})}{\sum_i (x_i - \bar{x})^2}$$

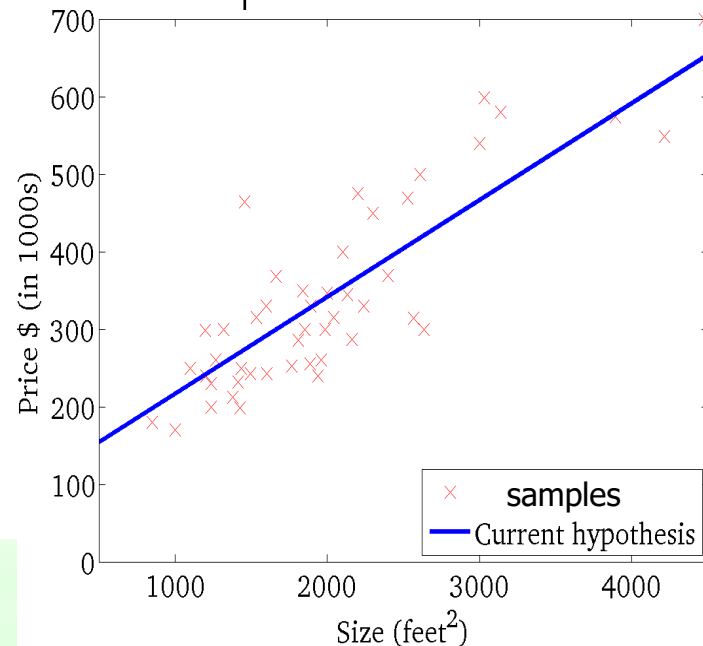
$$\beta_0 = \bar{y} - \beta_1 \bar{x}$$

- where $\bar{y} = \frac{1}{n} \sum_i y_i$ and $\bar{x} = \frac{1}{n} \sum_i x_i$ are the sample means

Simple linear regression: example



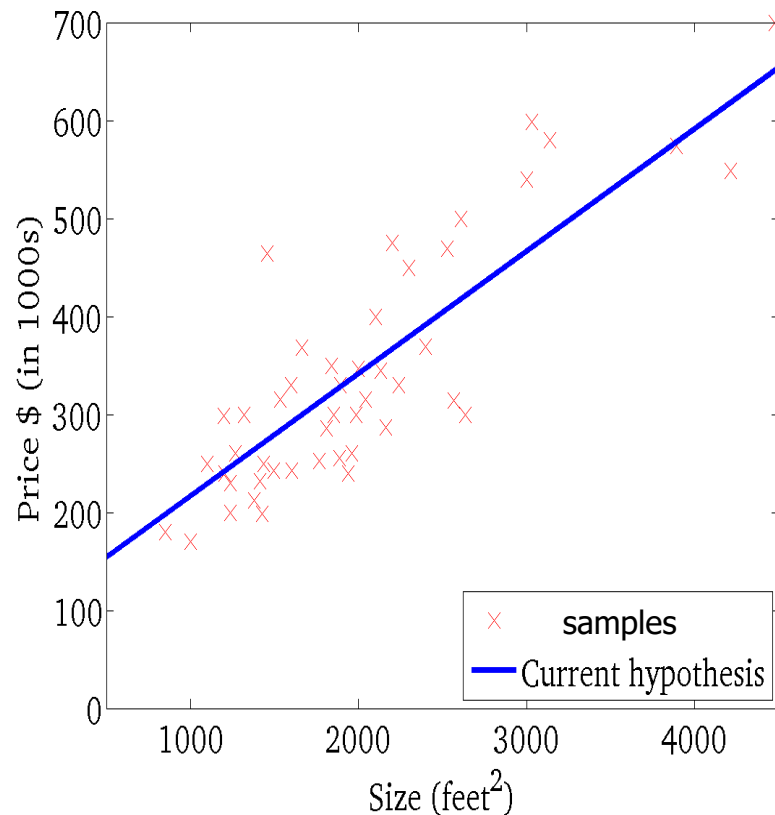
Size in feet ²	Price (\$) in 1000's
2104	460
1416	232
1534	315
852	178
...	...



- Goal of a **real estate agency**
 - Estimate the selling price of a home based on the value of size in square feet
- Simple linear regression finds a **linear model** of the problem
 - $x = \text{Size in feet}^2$
 - $y = \text{Price (\$) in 1000's}$

$$y = \beta_0 + \beta_1 x$$

Simple linear regression: example



- β_0 : The **intercept** represents the estimated value of y when x assumes 0
 - No house had 0 square feet, but β_0 is the portion of house price not explained by square feet
- β_1 : the **slope** measures the estimated change in the y value as for every one-unit change in x
 - The average value of a square foot of size

Multiple linear regression

$$y = f(\mathbf{x}) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \dots + \beta_n x_n + \xi$$

- **Dependant variable** (y): the single variable being explained/ predicted by the regression model
- **Independent or explanatory variables** (x_j): The variables used to predict/explain the dependant variable
- **Coefficients** (β_i): values, computed by the regression task, reflecting explanatory to dependent variable relationships
- **Residuals** (ξ): the portion of the dependent variable that is not explained by the model
 - The model performs under or over predictions

Interpreting regression coefficients



- Uncorrelated predictors
 - Each coefficient can be estimated and tested separately
 - Interpretation: a unit change in x_i is associated with a β_i change in y , while all the other variables stay fixed
 - β_i represents the average effect on y of a one unit increase in x_i , holding all other predictors fixed
- Correlation among predictors cause problems
 - The variance of all coefficients tends to increase, sometimes dramatically
 - Interpretations become complex: when x_j changes, everything else changes
- The claim of causality should be avoided for the observational data

Feature selection



- In case of a high dimensional data set, in terms of number of dependent variables, **some** of the **variables** might provide **redundant information**.
- Feature selection and removal (correlation-based approach)
 - simplifying the model computation
 - improving the model performance
 - Enhancing the model interpretation (i.e., better explainability of the dependent variables)
- Variable/feature selection
 - Driven by the business understanding and domain knowledge
 - Feature selection based on correlation test
 - Features highly-correlated with other attributes could be discarded from the analysis
 - having dependence or association in any statistical relationship, whether causal or not

Polynomial regression



- The polynomial models can be used in those situations where the **relationship** between dependent and explanatory variables is **curvilinear**.
- Polynomial regression consists of:
 - Computing new **features** that are power functions of the input features
 - Applying **linear** regression on these new features

$$y = \beta + \beta_1 x + \beta_2 x^2 + \varepsilon$$

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{11} x_1^2 + \beta_{22} x_2^2 + \beta_{12} x_1 x_2 + \varepsilon$$

- The above models are also linear (i.e., A model is linear when it is linear in parameters)
 - They are the second order polynomials in one and two variables respectively.
- Sometimes a nonlinear relationship in a small range of explanatory variables can also be modeled by polynomials.

Polynomial model in one variable

- The k^{th} order polynomial model in one variable is given by

$$y = \beta_0 + \beta_1 x + \beta_2 x^2 + \cdots + \beta_k x^k + \varepsilon$$

- It is included in the linear regression model below

$$y = X\beta + \varepsilon$$

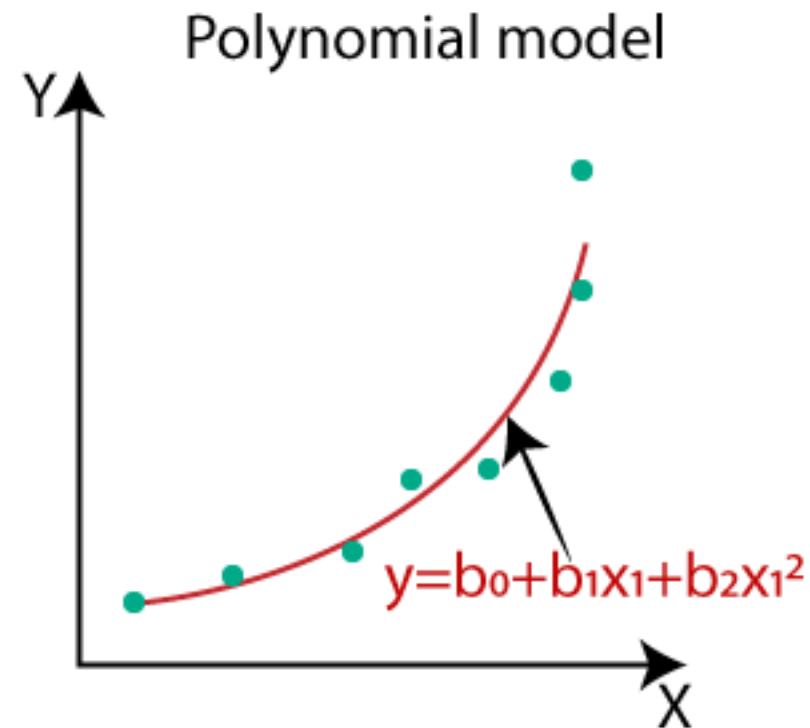
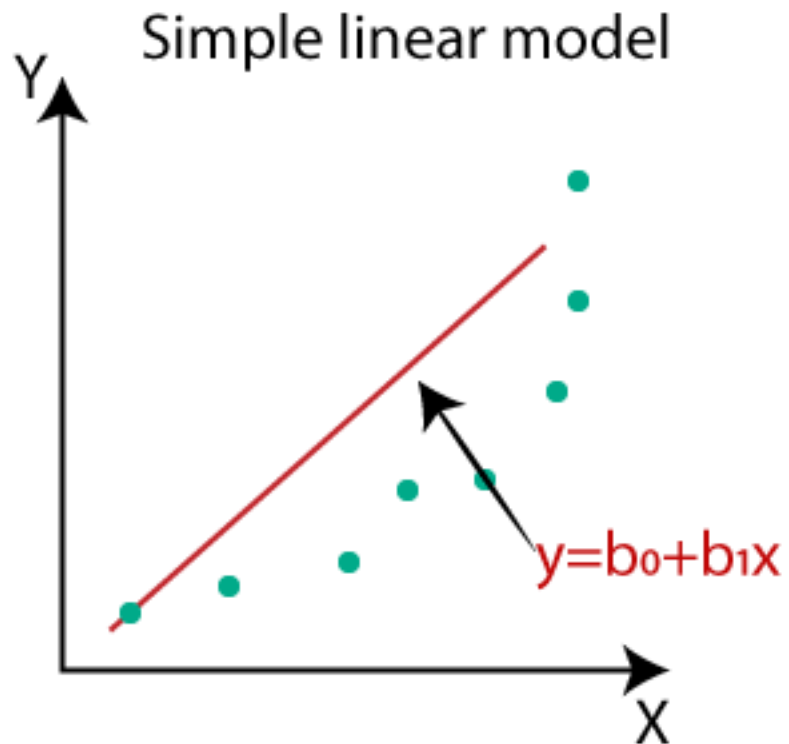
- Techniques for fitting linear model can be used for fitting the polynomial regression model
- For example, $y = \beta_0 + \beta_1 x + \beta_2 x^2$
 - Is a polynomial regression model in one variable and is called as **second order model** or **quadratic model**, where the coefficients
 - β_1 is the linear effect parameter
 - β_2 is the quadratic effect parameter
- The polynomial models can be used to approximate a complex nonlinear relationship

Polynomial model in one variable



Example.

Second order model or quadratic model





Polynomial regression: considerations in case of one variable

- Order of the model
 - Keep the order of the polynomial model as low as possible
 - Up to the **second order** polynomial
 - If necessary, you should apply some **data transformations**
 - Arbitrary fitting of higher order polynomials can be a serious abuse of regression analysis.
 - Data overfitting issue
- Different model building strategies do not necessarily lead to the same model
 - **Forward selection procedure:** to successively fit the models in increasing order and test the significance of regression coefficients at each step of model fitting.
 - Keep the order increasing until t-test for the highest order term is nonsignificant
 - The significance of highest order term is tested through the null hypothesis
 - **Backward elimination:** to fit the appropriate highest order model and then delete terms one at a time starting with highest order. This is continued until the highest order remaining term has a significant t-test
- The first and second order polynomials are mostly used in practice.

Polynomial models in two or more variables



- The techniques of fitting of polynomial model in one variable can be extended to fitting of polynomial models in two or more variables.
- A second order polynomial is more used in practice and its model is specified by

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{11} x_1^2 + \beta_{22} x_2^2 + \beta_{12} x_1 x_2 + \varepsilon$$

- This is also called **response surface**.

Strong and weak points of Polynomial Regression



- Advantages of using Polynomial Regression:
 - **Broad range of function** can be fit under it.
 - Polynomial basically fits **wide range of curvature**.
 - Polynomial usually provides the **best approximation** of the relationship between dependent and independent variable.
- Disadvantages of using Polynomial Regression
 - They are **too sensitive to the outliers**.
 - The presence of a few outliers in the data can seriously affect the results of a nonlinear analysis.
 - Higher polynomial degree means **higher flexibility** of your model, but also **data overfitting**
 - Overfitting occurs in those cases when you have a few samples and a model that has high flexibility
 - It is always possible for a polynomial of order $(n-1)$ to pass through n points so that a polynomial of sufficiently high degree can always be found that provides a “good” fit to the data.
 - Those models **never enhance the understanding** of the unknown function and they are **never good predictors**.

To avoid data overfitting



- Use more training data (if possible)
- Use lower model complexity
- Use regularization techniques
 - e.g., Ridge and Lasso

RIDGE and LASSO



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RIDGE and LASSO



- Regression analysis methods that perform both **variable selection** and **regularization** in order to enhance the prediction **accuracy** and **interpretability** of the statistical model it produces.
- Useful **to reduce model complexity** and **prevent overfitting** when
 - The number of variables describing each observation exceeds the number of observations
 - The number of variables does not exceed the number of observations, but the learned model suffers from poor generalization.
- Techniques of training a linear regression (or a linear regression with polynomial features)
 - They try to assign values **closer to zero (RIDGE) or zero (LASSO)** to the coefficients assigned to features that are not useful for the regression
 - The effect is the **decreasing of the complexity of the model**

Regularization: RIDGE and LASSO



Cost function

Linear regression

$$RSS = \sum_i (y_i - \hat{y}_i)^2 = \sum_i \left(y_i - \beta_0 - \sum_{j=1}^p \beta_j x_{ij} \right)^2$$

Ridge regression

$$RSS + \lambda \sum_{j=1}^p \beta_j^2$$

Lasso regression

$$RSS + \lambda \sum_{j=1}^p |\beta_j|$$

Penalty term $\lambda \rightarrow$ amount of shrinkage (or constraint)

Regularization



Ridge regression

- It adds L2 as the penalty
- L2 is the sum of the square of the magnitude of beta coefficients

$$RSS + \lambda \sum_{j=1}^p \beta_j^2$$

This is equivalent to minimizing the RSS under the condition

$$\text{For } c > 0, \sum_{j=1}^p \beta_j^2 < c$$

- Penalty term $\lambda \rightarrow$ amount of shrinkage (or constraint)
 - Regularizes the coefficients, penalizing coefficients taking large values

- LASSO means **Least Absolute Shrinkage and Selection Operator**
- Term coined by Robert Tibshirani in 1996, but it was originally introduced in geophysics literature 10 years before
- Lasso **regularization** was originally defined for **least squares**, but it is easily extended to a wide variety of statistical models in a straightforward fashion
 - E.g., generalized linear models
- The Lasso's **variable selection** relies on the form of the **constraint**
 - It forces the sum of the absolute value of the regression coefficients to be less than a fixed constraint, which forces some coefficients to be set to zero
 - The selected model is simpler since it does not include coefficients set to zero.
- It is similar to RIDGE regression but usually identifier a simpler model
 - **RIDGE** simplifies the model **by shrinking the size of some coefficients**, while **LASSO sets some coefficients to zero**.

Regularization



Lasso regression

- It adds L1 the penalty
- L1 is the sum of the absolute value of the beta coefficients

$$RSS + \lambda \sum_{j=1}^p |\beta_j|$$

This is equivalent to minimizing the RSS under the condition

$$\text{For } c > 0, \sum_{j=1}^p |\beta_j| < c$$

The regularization (L1) can lead to zero coefficients

- i.e., some of the features are completely neglected for the evaluation. It not only helps in reducing overfitting but also in feature selection

Support Vector Regression



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Simple linear regression vs Support Vector Regression



Recall that for linear regression, the parameters and the model can be derived by **minimizing the Residual sum of squares (RSS)**

$$\min RSS = \min \sum_i (y_i - \hat{y}_i)^2$$

We can instead be interested in reducing error to a certain degree

- errors within an acceptable range

Support Vector Regression

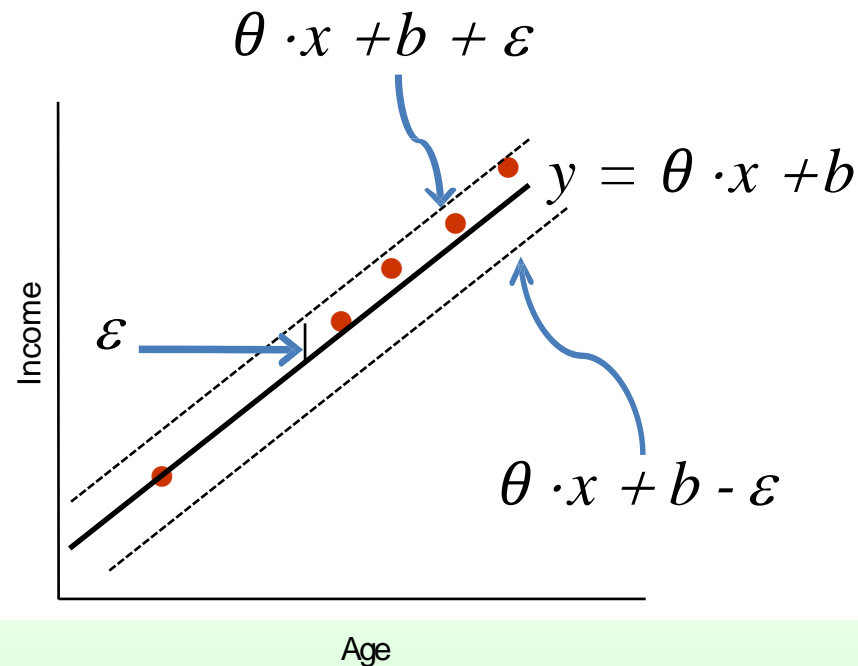
- define how much error is acceptable in our model
- find an appropriate hyperplane to fit the data

Support Vector Machine - Regression



- Find a function, $f(x)$, that performs a prediction of the target attribute y with a maximum error equal to ε

We do not care about errors as long as they are less than ε



Support Vector Regression: linear model



- The (training) problem can be formulated as a convex optimization problem

$$\min \frac{1}{2} \|\theta\|^2$$

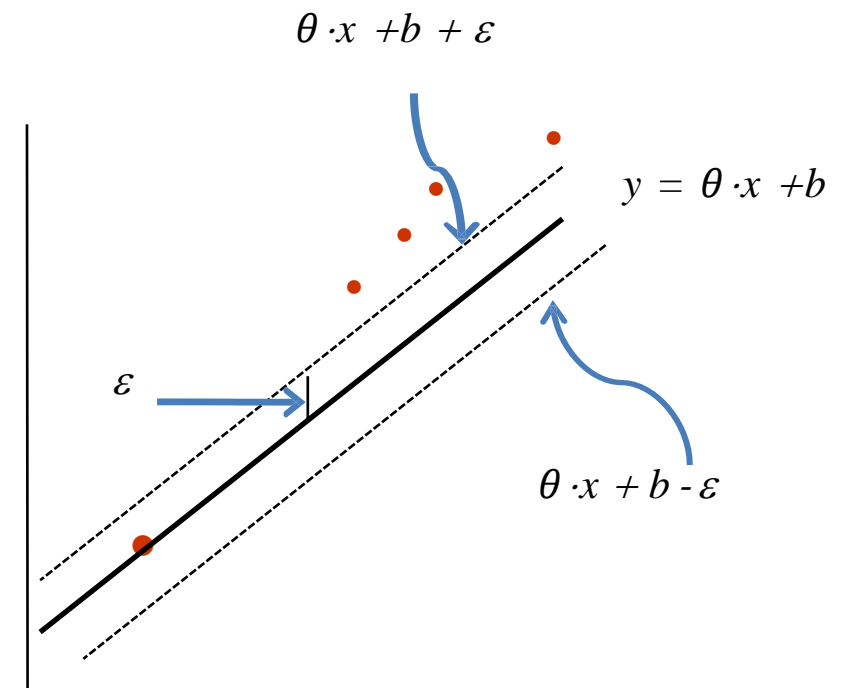
$$\text{s.t. } \begin{aligned} y^i - \theta \cdot x^i - b &\leq \varepsilon; \\ \theta \cdot x^i + b - y^i &\leq \varepsilon \end{aligned}$$

Constraints

y^i = value of the target attribute of the i^{th} training object

x^i = value of the predictive attributes of the i^{th} training object

θ and b = parameter of the regression model



Support Vector Regression: Soft margin



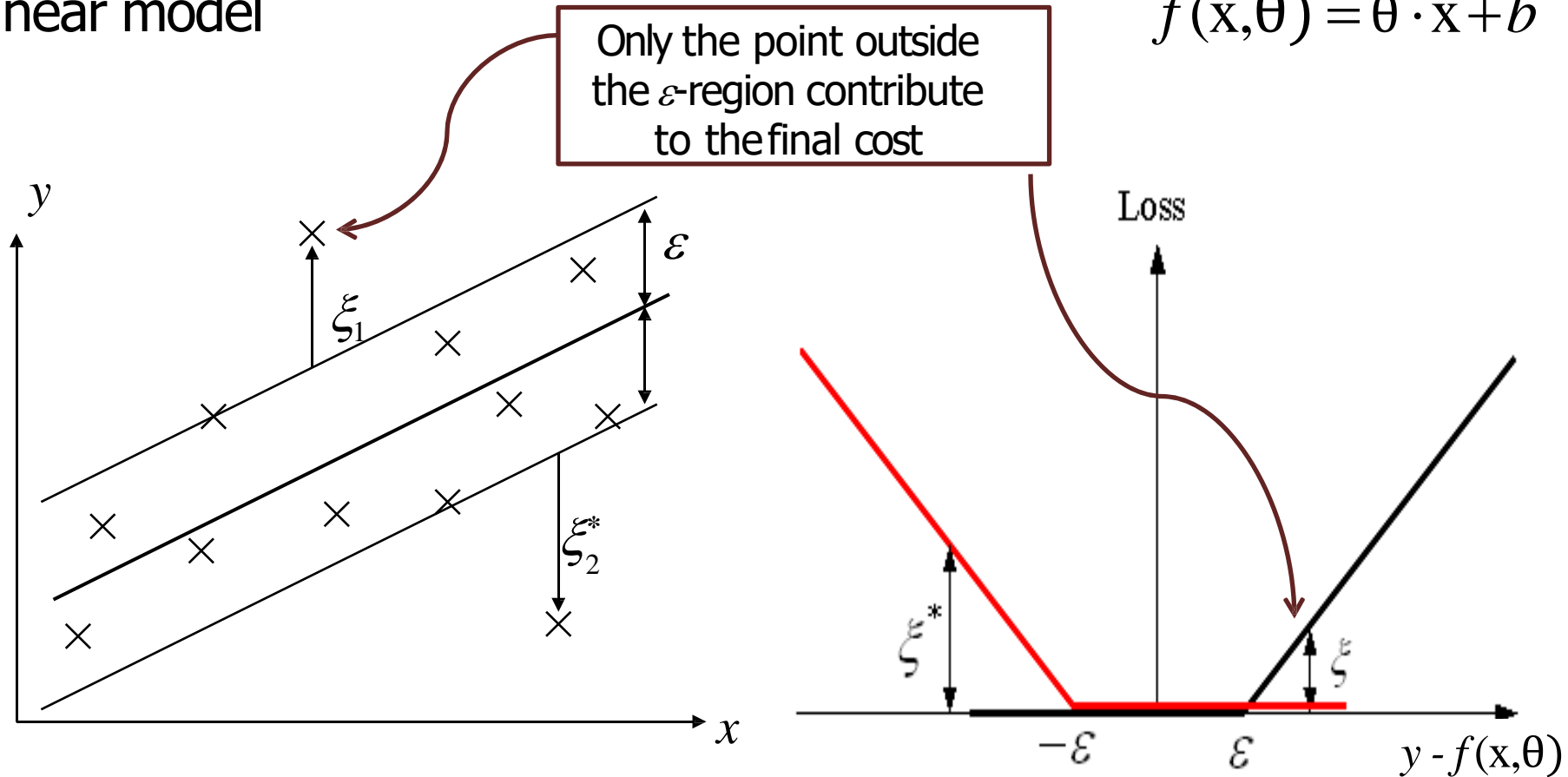
- Given a specific value of ε , the problem is not always feasible
- **Soft margin**
 - Reformulate the problem by considering the errors related to the predictions that do not satisfy the ε **maximum distance**

Support Vector Regression: Soft margin



Assume linear model

$$f(x, \theta) = \theta \cdot x + b$$



For any value that falls outside of ε , we can denote its deviation from the margin as ξ

Support Vector Regression: Soft margin



- The (training) problem can be formulated as a convex optimization problem

$$\min \frac{1}{2} \|\theta\|^2 + C \sum_{i=1}^m (\xi_i + \xi_i^*)$$

$$s.t. \quad y^i - \theta \cdot x^i - b \leq \varepsilon + \xi_i;$$

$$\theta \cdot x^i + b - y^i \leq \varepsilon + \xi_i^*$$

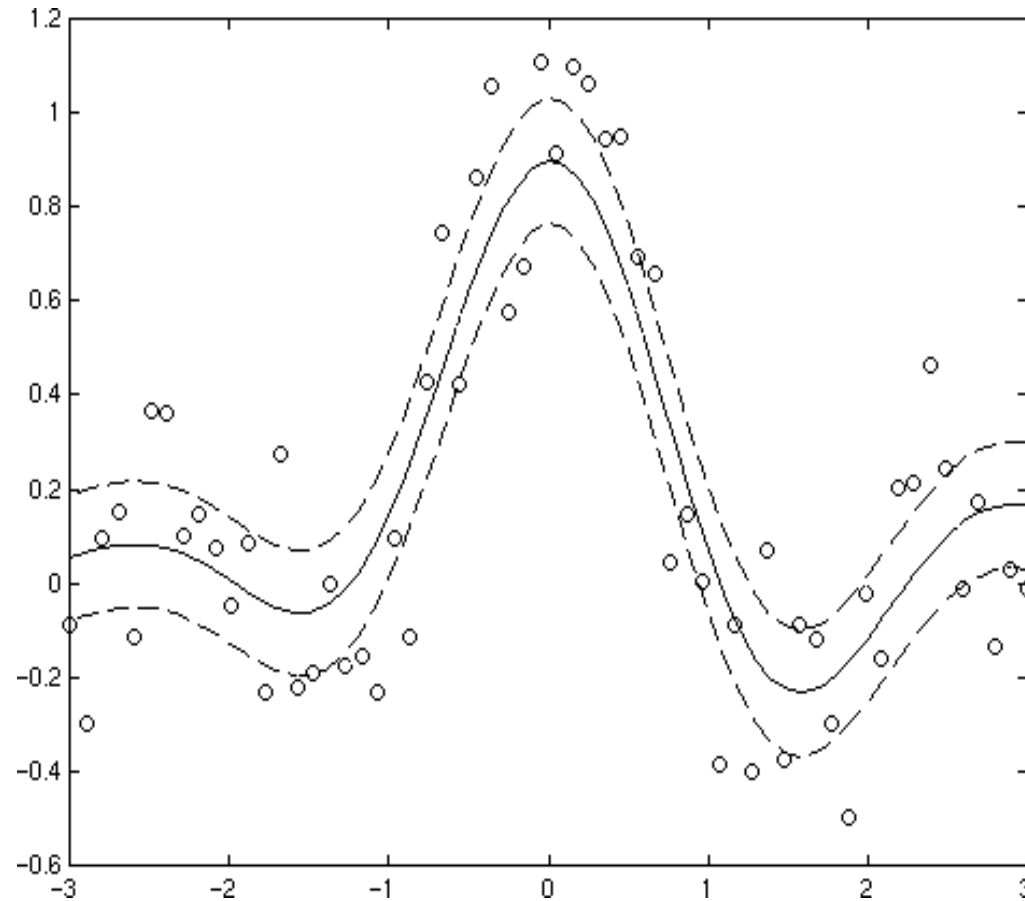
$$\xi_i, \xi_i^* \geq 0, i = 1, \dots, m$$

We minimize the deviation ξ from the margin

C: additional hyperparameter.

- As C increases, also the tolerance for points outside of ε decreases

How about a non-linear case?



Linear versus Non-linear SVR



- Map the original features into a higher order dimensional space
- Apply a kernel transformation
 - Polynomial
 - Gaussian radial
 - ...
- Transform the input data by means of the kernel function φ and then solve the previous problem

Linear versus Non-linear SVR



- φ maps the input data into a new dimensional space

$$\min \frac{1}{2} \|\theta\|^2 + C \sum_{i=1}^m (\xi_i + \xi_i^*)$$

$$s.t. \quad y^i - \theta \cdot \varphi(x^i) - b \leq \varepsilon + \xi_i;$$

$$\theta \cdot \varphi(x^i) + b - y^i \leq \varepsilon + \xi_i^*$$

$$\xi_i, \xi_i^* \geq 0, i = 1, \dots, m$$

Evaluating regression



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Evaluating regression



- Evaluation metrics for regression:
 - MAE (Mean Absolute Error)
 - MSE (Mean Squared Error)
 - RSE: Residual Standard Error
 - R^2
 - Adjusted R^2
- The evaluation is performed by comparing
 - y : the actual value (**ground truth**)
 - \hat{y} : the predicted value through the regression model

Evaluating regression



- **MAE (Mean Absolute Error)**

- the average vertical distance between each real value and the predicted one

$$MAE = \frac{1}{n} \sum_i |y_i - \hat{y}_i|$$

- **MSE (Mean Squared Error)**

- the average of the squares of the errors
- the average squared difference between the estimated values and the actual value.
- MSE tends to penalize less errors close to 0

$$MSE = \frac{1}{n} \sum_i (y_i - \hat{y}_i)^2$$

- **MAE and MSE always > 0**

- The lower the values of MAE and MSE the better the model
- It is mainly affected by the domains of data sample

Evaluating regression



- Overall accuracy of the model

- RSE: Residual Standard Error

$$RSE = \sqrt{\frac{1}{n-2}RSS} = \sqrt{\frac{1}{n-2} \sum_{i=1}^n (y_i - \hat{y}_i)^2}$$

- n is the number of samples
 - RSS is the residual sum of squares
 - RSE is always greater than 0
 - The lower the RSE value the better the regression model

Evaluating regression

- R^2 : R-squared measures the goodness of fit of a model
 - how well the regression predictions approximate the real data points.
 - It estimates a normalized error

$$R^2 = \frac{TSS - RSS}{TSS} = 1 - \frac{RSS}{TSS}$$

- RSS is the residual sum of squares

$$RSS = \sum_i (y_i - \hat{y}_i)^2$$

- TSS is the total sum of squares

with $\bar{y} = \frac{1}{n} \sum_i y_i$

$$TSS = \sum_i (y_i - \bar{y})^2$$

Evaluating regression: R^2



$$R^2 = 1 - \frac{RSS}{TSS} = 1 - FVU$$

$$= 1 - \frac{\sum_{i=1}^n (y_i - \hat{y}_i)^2}{\sum_{i=1}^n (y_i - \bar{y}_i)^2} = 1 - \frac{\frac{1}{n} \sum_{i=1}^n (y_i - \hat{y}_i)^2}{\frac{1}{n} \sum_{i=1}^n (y_i - \bar{y}_i)^2} = 1 - \frac{MSE}{\sigma^2}$$

- R^2 represents the proportion of variance of y explained by variation in x
 - FVU means the fraction of variance unexplained
 - Ratio between the unexplained variance (variance of the model's errors) and the total variance

Evaluating regression: R^2



- R^2 value
 - $R^2 = 1$
 - A perfect linear relationship between x and y
 - 100% of the Y variation is explained by variation in x
 - R^2 close to 1
 - A very good linear relationship between x and y
 - Good predictions
 - $0 < R^2 \ll 1$
 - Weaker linear relationship between x and y
 - A portion of the variation in y is not explained by variation in x
 - $R^2 = 0$
 - No linear relationship between x and y
 - The value of y does not depend on the value of x

Evaluating regression: R^2 adjusted

- Drawback of R^2
 - In the context of multiple linear regression, if new predictors (X_i) are added to the model, R^2 only increases or remains constant but it never decreases.
 - However, it is not always true that by increasing the complexity of regression model, the latter will be more accurate
- The Adjusted R-Squared is the modified form of R-Squared that has been adjusted to incorporate model's degree of freedom.
- It should be used to evaluate the quality of a multiple linear regression model

$$\bar{R}^2 = 1 - (1 - R^2) \frac{n-1}{n-p-1}$$

- p = number of explanatory variables
- n = number of samples
- The adjusted R-Squared only increases if the new term improves the model accuracy.